CERTAIN PROBLEMS OF HYDRODYNAMIC AND HYDROMAGNETIC STABILITY OF A CYLINDRICAL JET

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1. It is known that a quiescent fluid cylinder enclosed on the outside by another fluid is unstable in the presence of the surface tension forces [1]. In the paper [2] the results are presented of the investigation of the stability in the case when the cylinder moves along the axis with a velocity U = const with respect to the exterior medium. The examination is limited to the case of an axially symmetric disturbance.

Let us consider the stability of a cylindrical tangential discontinuity with surface tension with respect to an arbitrary disturbance of the form

$$f = f(r) \exp [i (kz + m\varphi - \omega t)] \qquad (1.1)$$

Inside the cylinder, for $r < a$, the quantities will be denoted by the subscript i ($v_{zi} = U = \mathrm{const}, p_i, \rho_i$), outside of the cylinder the subscript e ($v_{ze} = 0, p_e, \rho_e$) will be used. Proceeding from the equations for ideal hydrodynamics

$$\operatorname{div} \mathbf{v} = 0, \qquad \rho d\mathbf{v} / dt = - \nabla p \qquad (1.2)$$

for the disturbance (1.1) we obtain the dispersion equation

$$\rho_i\beta_i (kU - \omega)^2 + \rho_e\beta_e\omega^2 = \alpha \left[k^2 + \frac{m^2 - 1}{a^2}\right] \qquad \left(\beta_i = \frac{I_m (ka)}{kI_m' (ka)}, \quad \beta_e = -\frac{K_m (ka)}{kK_m' (ka)}\right) \qquad (1.3)$$

Here α is the coefficient of surface tension, $I_m(ka)$, $K_m(ka)$ are Bessel functions with imaginary arguments. The coefficients β_i and β_e are positive. The roots of the dispersion equation are

$$\omega = \frac{kU}{1 + A} \pm \left\{\frac{1}{\rho_i\beta_i (1 + A)}\left[\alpha \left(k^2 + \frac{m^2 - 1}{a^2}\right) - \frac{\rho_e\beta_e k^2U^2}{1 + A}\right]\right\}^{1/2} \qquad \left(A = \frac{\rho_e\beta_e}{\rho_i\beta_i}\right) \qquad (1.4)$$

In the particular case with $U = 0$ the dispersion equation (1.3) describes the stability of equilibrium of a fluid cylinder [1]. The form of Equation (1.4) with $U = 0$ is simplified. Because of the presence of surface tension, this column of fluid is unstable with respect to the disturbance of the type ($m = 0$) with wave length greater than the radius ($ka < 1$). The cylindrical tangential discontinuity without surface tension ($\alpha = 0$) is unstable with respect to the disturbance for arbitrary m and any wave length. Surface tension shows a stabilizing influence on short wave disturbances ($ka > 1$) and a destabilizing influence on long wave disturbances of the type ($m = 0$).

2. Let us consider hydromagnetic stability of an ideally conductive plasma jet submerged in a nonconductive fluid. Shafranov [3] investigated the stability of a plasma string retained by an exterior magnetic field. We will assume additionally that outside the string there exists a nonconducting fluid and the string moves with respect to the exterior fluid at a velocity U = const along its axis. For the undisturbed state we set

$$H_{\varphi i} = 0, \qquad H_{zi} = \text{const}, \qquad p_i = \text{const}, \qquad v_{zi} = U \qquad \text{for } r < a$$

$$H_{\varphi e} = H_0 a / r, \qquad H_{ze} = \text{const}, \qquad p_e = \text{const}, \qquad v_{ze} = 0 \qquad \text{for } r > a$$

Omitting the calculations analogous to those performed in [3] on the basis of the system of equations of ideal magneto-hydrodynamics

$$\operatorname{div} \mathbf{v} = 0, \quad \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}[\mathbf{j} \times \mathbf{H}], \quad \operatorname{div} \mathbf{H} = 0, \quad \frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{v} \times \mathbf{H}] \quad (r < a)$$

$$\operatorname{div} \mathbf{v} = 0, \qquad \rho \frac{d\mathbf{v}}{dt} = -\nabla p \qquad (r > a) \tag{2.1}$$

for the disturbance of the form (1.1) we obtain the dispersion equation

$$(kU - \omega)^2 + A\omega^2 = B + v_\text{н}^2 k^2 \qquad \text{for } a = 1 \tag{2.2}$$

where

$$A = -\frac{\rho_e I_m'(k) K_m(k)}{\rho_i I_m(k) K_m'(k)} > 0, \qquad B = \frac{H_0^2 k I_m'(k)}{4\pi\rho_i I_m(k)} \left\{ \frac{(m + kh_e)^2}{\varphi} - 1 \right\}$$

$$\varphi = m + \frac{kK_{m-1}(k)}{K_m(k)}, \qquad v_\text{н}^2 = \frac{H_{zi}^2}{4\pi\rho_i}, \qquad h_e = \frac{H_{ze}}{H_0}$$

The condition of stability has the form

$$B + k^2 v_\text{н}^2 - \beta k^2 U^2 > 0, \qquad \beta = A / (1 + A) \tag{2.3}$$

If the plasma cylinder rests in a nonconductive gas (U = 0), from (2.3) follows the condition of stability obtained by Shafranov. It is evident from (2.3) that the plasma jet moving with velocity U in an exterior gas, is less stable than the quiescent plasma string (the term $\beta k^2 U^2$ worsens stability). The condition of stability (2.3) can be interpreted in the following sense: the magnetic field H_{zi} stabilizes the tangential discontinuity of the velocity, for $v_\text{н}^2 > \beta U^2$ stabilizing of the tangential discontinuity by the magnetic field is possible.

We will show that the criterion of stability of the jet essentially depends on the distribution of the current density in the cross-section of the jet. With this in mind we consider the following jet model (a jet with a homogeneous axial current and a longitudinal field):

$$H_{\varphi i} = H_0 \frac{r}{a}, \qquad H_{zi} = \text{const}, \qquad v_{zi} = U; \qquad H_{\varphi e} = H_0 \frac{a}{r}, \qquad H_{ze} = H_{zi}, \qquad v_{ze} = 0$$

The pressure p_i falls off according to a parabolic law to the value p_e at $r = a$. On the boundary of the jet the pressure and the magnetic field are continuous.

Among the disturbances of the form (1.1) there exist those, that do not distort the magnetic field. In fact, from the condition of frozen flow it follows that

$$(kU - \omega)\, \mathbf{H}_i^{(1)} = \left(\frac{m}{r} H_{\varphi i} + kH_{zi} \right) \mathbf{v}_i^{(1)}$$

With $(m / r) H_{\varphi i} + kH_{zi} = 0$ only the velocity and pressure disturbances are different from zero. Since the magnetic field is not disturbed, it does not influence the development of the velocity and pressure disturbances. For such disturbances we obtain the dispersion equation of ordinary hydrodynamics (Equation (1.3) for α = 0). The plasma jet will be unstable in the presence of any velocity U . Thus, the criteria for stability of the plasma jet strongly depend on the distribution of the current density through the cross-section of the jet. The surface currents and the discontinuities of the magnetic field on the boundary of the jet stabilize the tangential discontinuity of the velocity. In the case of a continuous distribution of the current density through the cross-section of the jet the plasma jet may show itself to be unstable at any magnitude of the velocity jump.

3. We limit ourselves to the investigation of the following model of a cylindrical tangential discontinuity in an ideally conducting fluid:

$$H_{\varphi i} = H_0 \frac{r}{a}, \quad H_{zi} = \text{const}, \quad v_{zi} = U, \quad H_{\varphi e} = H_0 \frac{r}{a}, \quad H_{ze} = H_{zi}, \quad v_{ze} = 0$$

(the longitudinal field H_z and the current density j_z are homogeneous). For a disturbance of the form (1.1) from the conditions of frozen flow we obtain

$$(kU - \omega)\, \mathbf{H}_i^{(1)} = \left(\frac{m}{r} H_{\varphi i} + kH_{zi}\right) \mathbf{v}_i^{(1)}, \qquad -\omega \mathbf{H}_e^{(1)} = \left(\frac{m}{r} H_{\varphi e} + kH_{ze}\right) \mathbf{v}_e^{(1)}$$

If $m / rH_\varphi + kH_z = 0$, then the magnetic field is not disturbed and the criterion of stability of the tangential discontinuity does not depend on the magnetic field (again we obtain the dispersion equation (1.3) for $\alpha = 0$).

It appears that the magnetic field component (H_φ) transverse to the velocity, significantly weakens the stabilizing effect of the magnetic field. In fact, if $H_\varphi = 0$, then for $k \neq 0$ the magnetic field always deforms and effectively stabilizes the tangential discontinuity of the velocity. With $H_\varphi = 0$ we obtain the dispersion equation

$$\rho_i \beta_i (kU - \omega)^2 + \rho_e \beta_e \omega^2 = k^2 \frac{\beta_i H_{zi}^2 + \beta_e H_{ze}^2}{4\pi} \tag{3.1}$$

The condition of stability of the tangential discontinuity has the form

$$\beta_i H_{zi}^2 + \beta_e H_{ze}^2 \geqslant 4\pi \frac{\rho_i \beta_i \rho_e \beta_e}{\rho_i \beta_i + \rho_e \beta_e} U^2 \tag{3.2}$$

The results of this Section are in agreement with those obtained by Syrovatskii with respect to the stability of plane tangential discontinuities [4].

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Editorial Note.

*) H. Lamb, Hydrodynamics, Cambridge University Press, 1947.

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